The existence of light-like homogeneous geodesics in homogeneous Lorentzian manifolds

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Abstract

In previous papers, a fundamental affine method for studying homogeneous geodesics was developed. Using this method and elementary differential topology it was proved that any homogeneous affine manifold and in particular any homogeneous pseudo-Riemannian manifold admits a homogeneous geodesic through arbitrary point. In the present paper this affine method is refined and adapted to the pseudo-Riemannian case. Using this method and elementary topology it is proved that any homogeneous Lorentzian manifold of even dimension admits a light-like homogeneous geodesic. The method is illustrated in detail with an example of the Lie group of dimension 3 with an invariant metric, which does not admit any light-like homogeneous geodesic.

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1 Introduction

Let M be a pseudo-Riemannian manifold. If there is a connected Lie group $G \subset I_0(M)$ which acts transitively on M as a group of isometries, then M is called a *homogeneous pseudo-Riemannian manifold*. It can be naturally identified with the *pseudo-Riemannian homogeneous space* $(G/H, g)$, where H is the isotropy group of the origin $p \in M$.

If the metric g is positive definite, then $(G/H, g)$ is always a *reductive homogeneous space*: We denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H respectively and consider the adjoint representation $\mathrm{Ad}\colon H \times \mathfrak{g} \to \mathfrak{g}$ of H on $\mathfrak{g}$. There exists the *reductive decomposition* of the form $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ where $\mathfrak{m} \subset \mathfrak{g}$ is a vector subspace such that $\mathrm{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m}$. For a fixed reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ there is the natural identification of $\mathfrak{m} \subset \mathfrak{g} = T_eG$ with the tangent space T_pM via the projection $\pi\colon G \to G/H = M$. Using this natural identification and the scalar product g_p on T_pM, we obtain the invariant scalar product $\langle\,,\rangle$ on $\mathfrak{m}$.

If the metric g is indefinite, the reductive decomposition may not exist (see for instance [7] or [8] for examples of nonreductive pseudo-Riemannian homogeneous spaces). In such a case, we can study the manifold M using a more fundamental affine method, which was proposed in [6] and [4]. It is based on the well known fact that homogeneous pseudo-Riemannian manifold M with the origin p admits $n = \dim M$ Killing vector fields which are linearly independent at each point of some neighbourhood of p.

A geodesic $\gamma(s)$ through the point p is *homogeneous* if it is an orbit of a one-parameter group of isometries. More explicitly, if s is an affine parameter and $\gamma(s)$ is defined in an open interval J, there exists a diffeomorphism $s = \varphi(t)$ between the real line and the open interval J and a nonzero vector $X \in \mathfrak{g}$ such that $\gamma(\varphi(t)) = \exp(tX)(p)$ for all $t \in \mathbb{R}$. The vector X is called *geodesic vector*. The diffeomorphism $\varphi(t)$ may be nontrivial only for null curves in a properly pseudo-Riemannian manifold.

In the reductive case, geodesic vectors are characterized by the following *geodesic lemma* (see [10] for the Riemannian version, [7] for the first formulation in the pseudo-Riemannian case and [5] for the complete mathematical proof).

Lemma 1 *Let $X \in \mathfrak{g}$. Then the curve $\gamma(t) = \exp(tX)(p)$ is geodesic with respect to some parameter s if and only if*

$$\langle [X, Z]_{\mathfrak{m}}, X_{\mathfrak{m}} \rangle \quad = \quad k \langle X_{\mathfrak{m}}, Z \rangle$$

for all $Z \in \mathfrak{m}$ and for some constant $k \in \mathbb{R}$. If $k = 0$, then t is an affine parameter for this geodesic. If $k \neq 0$, then $s = e^{-kt}$ is an affine parameter for the geodesic. The second case can occur only if the curve $\gamma(t)$ is a null curve in a properly pseudo-Riemannian space.

In the paper [9], it was proved that any homogeneous Riemannian manifold admits a homogeneous geodesic through the origin. The generalization to the pseudo-Riemannian (reductive and nonreductive) case was obtained in [3] in the framework of a more general result, which says that any homogeneous affine manifold (M, ∇) admits a homogeneous geodesic through the origin. Here the affine method from [6] and [4], based on the study of integral curves of Killing vector fields, was used. The proof is using differential topology, namely the degree of a smooth mapping $S^n \to S^n$ without fixed points.

A homogeneous pseudo-Riemannian manifold all of whose geodesics are homogeneous is called a pseudo-Riemannian *g.o. manifold* or *g.o. space*. Their analogues with noncompact isotropy group are *almost g.o. spaces*. For many results and further references on homogeneous geodesics in the reductive case see for example the survey paper [2].

In pseudo-Riemannian geometry, null homogeneous geodesics are of particular interest. In [7] and [11], plane-wave limits (Penrose limits) of homogeneous spacetimes along light-like homogeneous geodesics were studied. However, it was not known whether any homogeneous pseudo-Riemannian or Lorentzian manifold admits a null homogeneous geodesic.

In [1], an example of a 3-dimensional Lie group with an invariant Lorentzian metric which does not admit light-like homogeneous geodesic was described. Here the standard geodesic lemma was used, because the example is reductive.

In the present paper, the affine method used in [3], [4] and [6] for the study of homogeneous affine manifolds is adapted to the pseudo-Riemannian case. As the main result it is shown that any Lorentzian homogeneous manifold of even dimension admits a light-like homogeneous geodesic through the origin. The calculation is particularly easy in the case of a Lie group $G = M$ with

a left-invariant metric. As an illustration, the method is applied on an example of a Lie group from [1].

2 The main result

Let (M, g) be a homogeneous pseudo-Riemannian manifold of dimension n, let G be a group of isometries acting transitively on M and let $p \in M$. Let ∇ be the induced pseudo-Riemannian connection on M. It is well known that there exist n Killing vector fields $K_1, \ldots, K_n$ on M which are linearly independent at each point of some neighbourhood U of p. Let $B = \{K_1(p), \ldots, K_n(p)\}$ be the basis of the tangent space T_pM. Any tangent vector $X \in T_pM$ has coordinates $(x^1, \ldots, x^n)$ with respect to the basis B and it determines the Killing vector field $X^* = x^1 K_1 + \ldots + x^n K_n$ and the integral curve γ_X of X^* through p. The following Proposition is a standard one.

Proposition 2 *Let $\phi_X(t)$ be the 1-parameter group of isometries corresponding to the Killing vector field X^*. For all $t \in \mathbb{R}$, it holds*

$$\phi_X(t)(p) = \gamma_X(t), \qquad \phi_X(t)_*(X^*_p) = X^*_{\gamma_X(t)}.$$

It is well known that the covariant derivative $\nabla_{X^*} X^*$ depends only on the values of the vector field X^* along the curve $\gamma_X(t)$. From the invariance of the metric g and the connection ∇ with respect to the group G, we obtain the following:

Proposition 3 *Along the curve $\gamma_X(t)$, it holds for all $t \in \mathbb{R}$*

$$\begin{aligned} g_p(X^*, X^*) &= g_{\gamma_X(t)}(X^*_{\gamma_X(t)}, X^*_{\gamma_X(t)}), \\ \phi_X(t)_*(\nabla_{X^*} X^*\big|_p) &= \nabla_{X^*} X^*\big|_{\gamma_X(t)}. \end{aligned}$$

Now we formulate the crucial feature.

Proposition 4 *Let (M, g) be a homogeneous Lorentzian manifold, $p \in M$ and $X \in T_pM$. Then, along the curve $\gamma_X(t)$, it holds*

$$\nabla_{X^*} X^*\big|_{\gamma_X(t)} \in (X^*_{\gamma_X(t)})^\perp.$$

Proof. We use the basic property $\nabla g = 0$ in the form

$$\nabla_{X^*} g(X^*, X^*) = 2g(\nabla_{X^*} X^*, X^*). \tag{1}$$

According to Proposition 3, the function $g(X^*, X^*)$ is constant along $\gamma_X(t)$. Hence, the left-hand side of the equality (1) is zero and the right-hand side gives the statement. □

Theorem 5 *Let (M, g) be a homogeneous Lorentzian manifold of even dimension n and let $p \in M$. There exist a light-like vector $X \in T_pM$ such that along the integral curve $\gamma_X(t)$ of the Killing vector field X^* it holds*

$$\nabla_{X^*} X^*\big|_{\gamma_X(t)} = k \cdot X^*_{\gamma_X(t)}, \tag{2}$$

where $k \in \mathbb{R}$ is some constant.

Proof. Let us choose the Killing vector fields $K_1, \dots K_n$ such that the vectors $K_1(p), \dots, K_n(p)$ form a pseudo-orthonormal basis of T_pM with $K_n(p)$ timelike. Again, any airthmetic vector $x = (x^1, \dots, x^n) \in \mathbb{R}^n$ determines the Killing vector field $X^* = \sum_{i=1}^n x^i K_i$. Using the identification of x with X^*_p we identify $\mathbb{R}^n$ with T_pM. There is the natural scalar product on $\mathbb{R}^n$ which comes from the scalar product g_p on T_pM and this identification. Let us consider arithmetic vectors of the form $x = (\tilde{x}, 1)$, where $\tilde{x} \in S^{n-2} \subset \mathbb{R}^{n-1}$. For the corresponding vector field X^*, we have $g_p(X^*_p, X^*_p) = 0$ and the vectors $\tilde{x} \in S^{n-2}$ determine light-like directions in $\mathbb{R}^n \simeq T_pM$.

For each light-like vector $x = (\tilde{x}, 1) \in \mathbb{R}^n \simeq T_pM$, we denote $Y_x = \nabla_{X^*} X^* \big|_p$. With respect to the pseudo-orthonormal basis $B = \{K_1(p), \dots, K_n(p)\}$, we denote the components of the vector Y_x as $y(x) = (y^1, \dots, y^n)$. Using Proposition 4, we see that $y(x) \perp x$. We define the new vector t_x as

$$t_x \quad = \quad y(x) - y^n \cdot x.$$

Because x is light-like vector, it holds also $t_x \perp x$. For the components of t_x, we have $t_x = (\tilde{t}_x, 0)$, where $\tilde{t}_x \in \mathbb{R}^{n-1}$. We easily see that $\tilde{t}_x \perp \tilde{x}$, with respect to the positive scalar product on $\mathbb{R}^{n-1}$ which is the restriction of the indefinite scalar product on $\mathbb{R}^n$. The assignment $\tilde{x} \mapsto \tilde{t}_x$ defines a smooth tangent vector field on the sphere S^{n-2}. If n is even, then according to the well known topological theorems, this vector field must have a zero value. In other words, there exist a vector $\tilde{x} \in S^{n-2}$ such that for the corresponding vector $x = (\tilde{x}, 1)$ it holds $t_x = 0$. For this vector x, it holds $y(x) = k \cdot x$ and formula (2) for the corresponding Killing vector field X^* is satisfied at $t = 0$. Using Proposition 3, we obtain the formula for all $t \in \mathbb{R}$. $\square$

Corollary 6 *Let (M, g) be a homogeneous Lorentzian manifold of even dimension n and let $p \in M$. There exist a light-like homogeneous geodesic through p.*

Proof. We consider the vector $X \in T_pM$ which satisfies Theorem 5. The integral curve $\gamma_X(t)$ through p of the corresponding Killing vector field X^* is homogeneous geodesic. $\square$

3 Invariant metric on a Lie group

Let $M = G$ be a Lie group with a left-invariant pseudo-Rieamannian metric g acting on itself by left translations and let $p = e$ be the identity. For any tangent vector $X \in T_eM$ and the corresponding Killing vector field X^*, we consider the vector function $X^*_{\gamma_X(t)}$ along the integral curve $\gamma_X(t)$ through e. It can be uniquely extended to the left-invariant vector field L^X on G. Hence, along γ_X, we have

$$L^X_{\gamma_X(t)} = X^*_{\gamma_X(t)}. \tag{3}$$

At general points $q \in G$, values of left-invariant vector field L_X do *not* coincide with the values of the Killing vector field X^*, which is *right-invariant*. However,

as we are interested in calculations along the curve $\gamma_X(t)$, we can work with respect to the moving frame of left-invariant vector fields and use formula (3).

Proposition 7 *Let $\{L_1, \dots, L_n\}$ be a left-invariant moving frame on a Lie group G with a left-invariant pseudo-Riemannian metric g and the induced pseudo-Riemannian connection ∇. Then it holds*

$$\nabla_{L_i} L_j = \sum_{k=1}^{n} \gamma_{ij}^k L_k, \qquad i, j = 1, \dots, n,$$

where γ_{ij}^k are constants.

Proof. It follows from the invariance of the affine connection ∇. □

Now we illustrate the affine method of the previous section with an example of the 3-dimensional Lie group $E(1,1)$ with an invariant Lorentzian metric which has no light-like homogeneous geodesic. We choose one of the examples described in the paper [1] by the standard method for reductive pseudo-Riemannian homogeneous manifolds and the geodesic lemma. We construct explicitly the vector field $\tilde{t}_x$, which has no zero value in this case.

The group $E(1,1)$ can be represented by the matrices of the form

$$\begin{pmatrix} e^{-w} & 0 & u \\ 0 & e^{w} & v \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence, the manifold $M = E(1,1)$ can be identified with the 3-space $\mathbb{R}^3[u, v, w]$. The left-invariant vector fields are $U = e^{-w}\partial_u$, $V = e^{w}\partial_v$, $W = \partial_w$. We choose the new moving frame $\{E_1, E_2, E_3\}$ given as

$$E_1 = U - V, \qquad E_2 = -W, \qquad E_3 = 1/2(U + V).$$

In this frame, we have the following rules for the Lie bracket

$$[E_1, E_3] = 0, \qquad [E_2, E_1] = 2E_3, \qquad [E_2, E_3] = 1/2 E_1.$$

We introduce the pseudo-Riemannian metric g such that the basis determined by the above frame at any point $p \in M$ is pseudo-orthonormal basis of T_pM with E_3 timelike (we keep the notation from [1] here).

It is straightforward to write down the above metric g in coordinates in the form

$$ds^2 \;\; = \;\; -\frac{1}{4}(3e^{2w}du^2 + 3e^{-2w}dv^2 + 10dudv - 4dw^2)$$

and to calculate the nonzero Christoffel symbols

$$\Gamma_{11}^3 = \frac{3}{4}e^{2w}, \quad \Gamma_{13}^1 = -\tfrac{9}{16}, \quad \Gamma_{13}^2 = \frac{15}{16}e^{2w},$$
$$\Gamma_{22}^3 = -\frac{3}{4}e^{-2w}, \quad \Gamma_{23}^2 = \tfrac{9}{16}, \quad \Gamma_{23}^1 = -\frac{15}{16}e^{-2w}.$$

However, we can write down the same information in the frame $\{E_1, E_2, E_3\}$. By definition, we have at any point $p \in M$

$$g(E_1, E_1) = g(E_2, E_2) = 1, \quad g(E_3, E_3) = -1, \quad g(E_i, E_j) = 0, \quad i \neq j.$$

By the straightforward calculations, we obtain nonzero covariant derivatives which satisfy Proposition 7:

$$\begin{aligned} \nabla_{E_1} E_2 &= -\frac{3}{4} E_3, \quad \nabla_{E_1} E_3 = -\frac{3}{4} E_2, \quad \nabla_{E_2} E_3 = \frac{5}{4} E_1, \\ \nabla_{E_2} E_1 &= \frac{5}{4} E_3, \quad \nabla_{E_3} E_1 = -\frac{3}{4} E_2, \quad \nabla_{E_3} E_2 = \frac{3}{4} E_1. \end{aligned} \tag{4}$$

We will perform all calculations in this moving frame, or with respect to the corresponding pseudo-orthonormal basis $B = \{E_1(e), E_2(e), E_3(e)\}$ of the tangent space $T_e M \simeq \mathbb{R}^3$ at the origin $e \in E(1, 1)$. Any arithmetic vector $x = (x^1, x^2, x^3) \in \mathbb{R}^3$ determines the left-invariant vector field

$$L^X = x^1 E_1 + x^2 E_2 + x^3 E_3.$$

We are interested in light-like vectors $X \in T_e M$, hence $x = (\sin(\varphi), \cos(\varphi), 1)$, $\tilde{x} = (\sin(\varphi), \cos(\varphi)) \in S^1$ for some $\varphi \in \mathbb{R}$. For the corresponding left-invariant vector field L^X we calculate using (4) the covariant derivative

$$\begin{aligned} \nabla_{L^X} L^X &= 2\cos(\varphi) E_1 - \frac{3}{2}\sin(\varphi) E_2 + \frac{1}{2}\sin(\varphi)\cos(\varphi) E_3, \\ y(x) &= \Big(2\cos(\varphi), -\frac{3}{2}\sin(\varphi), \frac{1}{2}\sin(\varphi)\cos(\varphi)\Big). \end{aligned}$$

We see immediately that $y(x) \perp x$. The projection t_x is

$$\begin{aligned} t_x &= y(x) - \frac{1}{2}\sin(\varphi)\cos(\varphi) \cdot x = \\ &= \Big(2\cos(\varphi) - \frac{1}{2}\sin^2(\varphi)\cos(\varphi), -\frac{3}{2}\sin(\varphi) - \frac{1}{2}\sin(\varphi)\cos^2(\varphi), 0\Big) = \\ &= \Big[2 - \frac{1}{2}\sin^2(\varphi)\Big] \cdot \Big(\cos(\varphi), -\sin(\varphi), 0\Big). \end{aligned}$$

We see that $t_x \perp x$ and $\tilde{t}_x \perp \tilde{x}$. Clearly, $\tilde{x} \mapsto \tilde{t}_x$ defines the smooth vector field on S^1, which is nonzero everywhere. Hence, there is not any vector $X \in T_e G$ which satisfies Theorem 5.

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